

Upgrading MLSI to LSI for reversible Markov chains

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Abstract

For reversible Markov chains on finite state spaces, we show that the modified log-Sobolev inequality (MLSI) can be upgraded to a log-Sobolev inequality (LSI) at the surprisingly low cost of degrading the associated constant by $\log(1/p)$, where p is the minimum non-zero transition probability. We illustrate this by providing the first log-Sobolev estimate for Zero-Range processes on arbitrary graphs. As another application, we determine the modified log-Sobolev constant of the Lamplighter chain on all bounded-degree graphs, and use it to provide negative answers to two open questions by Montenegro and Tetali (2006) and Hermon and Peres (2018). Our proof builds upon the ‘regularization trick’ recently introduced by the last two authors.

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1 Introduction

Functional inequalities constitute a powerful set of tools for the study of the *concentration of measure* phenomenon [24, 25]. On a conceptual level, those inequalities relate certain statistics of a measure (such as the variance and entropy) to the Dirichlet form of a Markov process that preserves the measure. Besides concentration, those inequalities are intimately connected to the rate of convergence to equilibrium of the considered Markov process. Perhaps the most popular functional inequalities are the *Poincaré Inequality* (PI) and the *log-Sobolev Inequality* (LSI), which were initially studied in the continuous setting. On discrete spaces, a variant called the *Modified log-Sobolev Inequality* (MLSI) has been put forward and exploited due to its connection with the entropic exponential ergodicity of the underlying semi-group. Informally, it is a standard fact (see for example [3]) that

$$\text{LSI} \Rightarrow \text{MLSI} \Rightarrow \text{PI},$$

and the aim of this paper is to investigate the reverse implications.

1.1 Setup

Throughout the paper, we consider an irreducible Markov generator Q on a finite state space \mathcal{X} , and we assume that Q is reversible with respect to a probability measure π . The associated *Dirichlet form* is given, for any observables $f, g: \mathcal{X} \rightarrow \mathbb{R}$, by the formula

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x, y \in \mathcal{X}} \pi(x) Q(x, y) (f(x) - f(y)) (g(x) - g(y)).$$

The *variance* and *entropy* of a function $f: \mathcal{X} \rightarrow (0, \infty)$ are defined as $\text{Var}(f) := \mathbf{E}[f^2] - \mathbf{E}^2[f]$ and $\text{Ent}(f) := \mathbf{E}[f \log f] - \mathbf{E}[f] \log \mathbf{E}[f]$, where $\mathbf{E}[\cdot]$ denotes expectation with respect to π and where ‘log’ stands for the natural logarithm. The *log-Sobolev* constant t_{LS} , the *modified log-Sobolev* constant t_{MLS} , and the *Poincaré* constant t_{REL} are then respectively defined as the optimal constants in the functional inequalities

$$\text{Ent}(f) \leq t\mathcal{E}(\sqrt{f}, \sqrt{f}); \tag{LSI}$$

$$\text{Ent}(f) \leq t\mathcal{E}(f, \log f); \tag{MLSI}$$

$$\text{Var}(f) \leq t\mathcal{E}(f, f), \tag{PI}$$

for all functions $f: \mathcal{X} \rightarrow (0, \infty)$. We refer the unfamiliar reader to the seminal papers [11, 23, 3], the book [27], or the more recent works [17, 29, 2, 7] for details about those fundamental constants and their many variants.

It is classical that (LSI), (MLSI), (PI) are decreasing in strength, in the exact sense that

$$\frac{t_{\text{REL}}}{2} \leq t_{\text{MLS}} \leq \frac{t_{\text{LS}}}{4}. \quad (1)$$

The leftmost inequality is obtained by a standard perturbation argument around constant functions, while the rightmost one is a direct consequence of the inequality

$$4\mathcal{E}(\sqrt{f}, \sqrt{f}) \leq \mathcal{E}(f, \log f), \quad (2)$$

for all $f: \mathcal{X} \rightarrow (0, \infty)$, which in turns follows from the easy bound $4(\sqrt{u}-\sqrt{v})^2 \leq (u-v) \log \frac{u}{v}$ for all $u, v > 0$. However, in contrast with what happens in the smooth setting of diffusions on manifolds, the functional inequality (2) can here *not* be reversed uniformly in f : indeed, as soon as $|\mathcal{X}| \geq 2$, the quantity $\mathcal{E}(f, \log f)$ can be made arbitrarily larger than $\mathcal{E}(\sqrt{f}, \sqrt{f})$ by simply increasing the value of f at a single point. This degeneracy is one of the many infamous consequences of the lack of a *chain rule* for Markov generators on discrete spaces [24]. It introduces a fundamental discrepancy between the inequalities (MLSI) and (LSI), which results in two very different meanings for their optimal constants: t_{MLS} measures the *entropy production* along the semi-group, while t_{LS} quantifies the much stronger *hypercontractivity*. As a result, the ratio $t_{\text{LS}}/t_{\text{MLS}}$ can be arbitrarily large even for two-point chains (see [3, Section 3]), and any general upper bound on it will inevitably have to depend on the data (Q, π) . Understanding *how* is the task we undertake here.

To the best of our knowledge, the only general upper bound available on the ratio $t_{\text{LS}}/t_{\text{MLS}}$ is the one resulting from the chain of inequalities

$$\frac{t_{\text{LS}}}{t_{\text{MLS}}} \leq \frac{2t_{\text{LS}}}{t_{\text{REL}}} \leq \frac{2}{1-2\pi_\star} \log\left(\frac{1}{\pi_\star} - 1\right), \quad (3)$$

where $\pi_\star := \min_{x \in \mathcal{X}} \pi(x)$. The first inequality follows from (1), and the second from a direct comparison with the Dirichlet form of the trivial chain $Q(x, y) = \pi(y)$, whose log-Sobolev constant is explicit (see, e.g. [11]). Note that this ‘extreme’ chain, which mixes in a single jump, actually saturates both inequalities in (3). However, a quick look at more reasonable examples should convince the reader that the ratio $t_{\text{LS}}/t_{\text{MLS}}$ is typically much smaller than $\log(1/\pi_\star)$ in practice, and this observation was the motivation behind the present paper.

1.2 Main result

Writing $E := \{(x, y) \in \mathcal{X}^2: Q(x, y) > 0\}$ for the (symmetric) set of allowed transitions and $Q(x) := \sum_{y \in \mathcal{X} \setminus \{x\}} Q(x, y)$ for the total jump rate at $x \in \mathcal{X}$, we define the *sparsity* parameter

$$p := \min_{(x,y) \in E} \frac{Q(x,y)}{Q(x) \vee Q(y,x)}. \quad (4)$$

In particular, in the standard *stochastic case* where $Q(x) = 1$ for all $x \in \mathcal{X}$, this reduces to

$$p = \min_{(x,y) \in E} Q(x,y).$$

Our main result is that the lost equivalence between (MLSI) and (LSI) on discrete spaces can be restored at the surprisingly low cost $\log(1/p)$ only.

Theorem 1 (Upgrading MLSI to LSI). *For any reversible Markov generator, we have*

$$t_{\text{LS}} \leq 20 t_{\text{MLS}} \log\left(\frac{1}{p}\right).$$

The improvement over (3) can be considerable, since p is typically much larger than π_* . To appreciate this, consider the important case of simple random walk on a finite graph $G = (V_G, E_G)$: the classical estimate (3) predicts $t_{\text{LS}}/t_{\text{MLS}} = O(\log |V_G|)$, whereas our result gives $t_{\text{LS}}/t_{\text{MLS}} = O(\log d)$ where d denotes the maximum degree. As a concrete example, when G is the transposition walk on the symmetric group \mathfrak{S}_n , we obtain $t_{\text{LS}}/t_{\text{MLS}} = O(\log n)$ instead of $t_{\text{LS}}/t_{\text{MLS}} = O(n \log n)$. Incidentally, this example demonstrates that our result is sharp except for the value of the universal prefactor, which we did not try to optimize (see, e.g. [13]). On bounded-degree graphs, Theorem 1 shows that (MLSI) and (LSI) are actually equivalent, a new fact with surprising consequences (see Section 1.3 below).

Beyond the theoretical interest of a universal comparison between (MLSI) and (LSI), Theorem 1 can be used in practice to derive new, ready-to-use functional-analytic estimates. Obviously, this can be done in two different directions. First, we can produce lower bounds on t_{MLS} in situations where a lower bound on t_{LS} is available. In Section 1.3 below, we illustrate this by determining the modified log-Sobolev constant of the Lamplighter chain on all bounded-degree graphs, and we use this to provide negative answers to two open questions by Montenegro and Tetali and by Hermon and Peres. Conversely, there are several concrete classes of chains for which a (MLSI) was established by methods that do not carry over to (LSI). Examples include Bernoulli-Laplace models [5], Zero-Range processes [21, 5],

or negatively-dependent measures on the Boolean hypercube [18, 8]. In such examples, using Theorem 1 to convert the known upper bound on t_{MLS} into a new one on t_{LS} is of practical interest for at least three reasons:

- (i) *Mixing times*: writing f_t for the density with respect to equilibrium of the Markov process at time t , the parameter t_{MLS} only controls the decay rate of the relative entropy $\text{Ent}(f_t)$, while t_{LS} controls the much stronger uniform norm $\|f_t - 1\|_\infty$ (see, e.g., [27]).
- (ii) *Isoperimetry*: by specializing (LSI) to $\{0, 1\}$ -valued functions, the constant t_{LS} captures *small-set expansion*, which has numerous applications (see, e.g., [14]).
- (iii) *Robustness*: unlike t_{MLS} , an estimate on t_{LS} can be transferred to other chains using the classical and well-developed comparison theory for Dirichlet forms (see, e.g., [12]).

In Section 1.4 below, we will illustrate this by providing the very first log-Sobolev estimate for the Zero-Range Process with increasing rates on arbitrary graphs.

1.3 Application to Lamplighter chains

Fix a finite graph $G = (V_G, E_G)$ and imagine that each vertex is equipped with a lamp that can be either *off* or *on*. Now, consider a lamplighter performing a simple random walk on G and randomly switching the lamps off or on on his way. More formally, let us describe the state of the system by a pair (i, σ) where $i \in V$ represents the position of the lamplighter and $\sigma \in \{0, 1\}^V$ indicates which lamps are on. The *Lamplighter chain* on G is the continuous-time Markov chain on $\mathcal{X} := V \times \{0, 1\}^V$ whose generator acts on any $f: \mathcal{X} \rightarrow \mathbb{R}$ as follows:

$$(Qf)(i, \sigma) = \frac{1}{2} (f(i, \sigma^i) - f(i, \sigma)) + \frac{1}{2 \deg(i)} \sum_{j \sim i} (f(j, \sigma) - f(i, \sigma)).$$

In this formula, the notation $j \sim i$ means that $\{i, j\} \in E_G$, while σ^i denotes the configuration obtained from $\sigma \in \{0, 1\}^V$ by replacing σ_i with $1 - \sigma_i$. The behavior of this process is extremely rich and has drawn considerable interest across various fields, including graph theory, group theory, spectral theory, discrete functional analysis and probability. We refer the reader to the book [26, Chapter 19] and the references therein for a quick introduction.

The works [16, 28, 1, 22, 15] are all devoted to the fundamental question of relating the mixing properties of the Lamplighter chain to those of the underlying graph G . In particular, the relaxation time, the total-variation mixing time and the uniform mixing time

are now completely understood. This is also true for the log-Sobolev constant, at least on bounded-degree graphs. Specifically, the Lamplighter chain on G was shown in [1] to satisfy

$$t_{\text{LS}} \asymp_d \frac{|V_G|}{\gamma(G)}, \quad (5)$$

where $\gamma(G)$ denotes the spectral gap of G and $d = \max_{i \in V_G} \deg(i)$ the maximum degree. Here, the notation $a \asymp_d b$ means that the ratio a/b is bounded from above and below by positive numbers that depend only on d . However, in contrast with many other mixing parameters, nothing seems to be known about the modified log-Sobolev constant of Lamplighter chains, even on simple graphs such as the n -cycle \mathbb{Z}_n . Since the Lamplighter chain has sparsity $p = 1/(2d)$, our main result allows us to determine t_{MLS} on all bounded-degree graphs.

Corollary 1 (MLSI for the Lamplighter chain on bounded-degree graphs). *We have*

$$t_{\text{MLS}} \asymp_d \frac{|V_G|}{\gamma(G)}.$$

On the discrete torus $G = \mathbb{Z}_n^d$, we thus obtain $t_{\text{MLS}} = \Theta(n^{d+2})$ for any fixed dimension $d \geq 1$. Surprisingly, this is always much larger than the total-variation mixing time of the Lamplighter chain, which is known [16, 28] to be $t_{\text{MIX}} = \Theta(n^2)$ when $d = 1$, $t_{\text{MIX}} = \Theta(n^2 \log^2 n)$ when $d = 2$, and $t_{\text{MIX}} = \Theta(n^d \log n)$ for any fixed $d \geq 3$. Thus, Lamplighter chains on discrete tori constitute a simple family of counter-examples to the following classical question, which appears as Question 8.2 in the classical monograph [27] by Montenegro and Tetali.

Question 1 (Question 8.2 in [27]). *Is there a universal constant $c > 0$ such that $t_{\text{MLS}} \leq c t_{\text{MIX}}$, for all irreducible Markov chains?*

Very recently, Hermon and Peres [17, Question 7.1] proposed the following more reasonable conjecture, in which t_{MIX} is replaced by the larger relative-entropy mixing time t_{ENT} (see [15] or [17] for the precise definition).

Question 2 (Question 7.1 in [17]). *Is there a universal constant $c > 0$ such that $t_{\text{MLS}} \leq c t_{\text{ENT}}$, for all irreducible Markov chains?*

Since the Lamplighter chain on the n -cycle \mathbb{Z}_n is known to satisfy $t_{\text{ENT}} = \Theta(n^2 \log n)$ [15], the estimate $t_{\text{MLS}} = \Theta(n^3)$ given by Corollary 1 again provides a negative answer to this question. Finally, we note that our example also refutes Question 7.2 in the same paper, which asks for an even stronger upper bound on t_{MLS} .

1.4 Application to Zero-Range Processes

Introduced by Spitzer [30], the *Zero-Range Process* (ZRP) is a generic interacting particle system in which individual jumps occur at a rate that only depends on the current number of particles present at the source. The model is parameterized by the following ingredients:

- two integers $m, n \geq 1$ representing the number of particles and sites, respectively;
- a symmetric stochastic matrix $G = (G_{ij})_{1 \leq i, j \leq n}$ specifying the geometry;
- a function $r_i: \{1, 2, \dots\} \rightarrow (0, \infty)$ encoding the kinetics at each site $i \in [n]$.

The ZRP with these parameters is a continuous-time Markov chain on the state space

$$\mathcal{X} := \left\{ x = (x_1, \dots, x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^n x_i = m \right\}, \quad (6)$$

where x_i represents the number of particles at site i . The action of the generator is given by

$$(Qf)(x) := \sum_{1 \leq i, j \leq n} r_i(x_i) G_{ij} (f(x + \delta_j - \delta_i) - f(x)), \quad (7)$$

where $(\delta_1, \dots, \delta_n)$ denotes the canonical n -dimensional basis, and with the convention that $r_i(0) = 0$ for all $i \in [n]$ (no jumps from empty sites). In words, a site i with $k \geq 1$ particles expels a particle at rate $r_i(k)$, and the destination is chosen according to G . A natural choice for the latter is the transition matrix of simple random walk on a regular graph. In fact, the model is already interesting on the complete graph (the so-called *mean-field* case):

$$\forall (i, j) \in [n]^2, \quad G_{ij} = \frac{1}{n}. \quad (8)$$

Obtaining quantitative estimates on the convergence to equilibrium of the ZRP has been and continues to be a subject of active research, see e.g., [6, 20] in the mean-field case and [9, 10] on integer lattices. A standard assumption on the rate functions $(r_i)_{1 \leq i \leq n}$ is that their increments all lie in a fixed compact subset of $(0, \infty)$:

$$\delta \leq r_i(k+1) - r_i(k) \leq \Delta, \quad (9)$$

for some $\delta, \Delta > 0$ and every $i \in [n]$ and $k \in \mathbb{Z}_+$. Under this *weak interaction* condition, and in the *mean-field* setting (8), the ZRP was shown in [21] to satisfy the dimension-free MLSI

$$t_{\text{MLS}} \leq \frac{2\Delta}{\delta^2}.$$

Note that our sparsity parameter p here satisfies $p \geq \frac{\delta}{\Delta mn}$, because $Q(x) \leq \sum_{i=1}^n r_i(x_i) \leq \Delta m$ for all $x \in \mathcal{X}$ and $Q(x, y) \geq \delta/n$ for all $(x, y) \in E$. Thus, Theorem 1 produces the following estimate which, to the best of our knowledge, is the very first LSI for the mean-field ZRP.

Corollary 2 (LSI for mean-field ZRP). *Under assumptions (8) and (9), we have*

$$t_{\text{LS}} \leq \frac{40\Delta}{\delta^2} \log \left(\frac{\Delta mn}{\delta} \right),$$

for any choice of the dimension parameters n, m .

The dependency in n is optimal, as can be seen by investigating the case of a single particle ($m = 1$). As already mentioned, one of the advantages of t_{LS} over t_{MLS} is its robustness under comparison of Dirichlet forms. This is particularly true in the present setting, because it was shown in [19] that replacing a general symmetric stochastic matrix G by its mean-field version (8) can not increase the Dirichlet form of the ZRP by more than a factor $1/\gamma(G)$, where $\gamma(G)$ denotes the spectral gap of G . Consequently, our mean-field LSI estimate can be directly transferred to arbitrary geometries, yielding the following result. To the best of our knowledge, log-Sobolev estimates for the ZRP were so far restricted to lattices [9].

Corollary 3 (LSI for ZRP on arbitrary geometries). *Under Assumption (9), we have*

$$t_{\text{LS}} \leq \frac{40\Delta}{\gamma(G)\delta^2} \log \left(\frac{\Delta mn}{\delta} \right),$$

for any choice of the dimension parameters n, m and of the symmetric stochastic matrix G .

Remark 1 (Asymmetric geometries). *Interestingly, the symmetry of G is never actually used in [21, 19], so Corollary 3 generalizes to any stochastic matrix G as follows:*

$$t_{\text{LS}} \leq \frac{40\Delta}{\gamma(G)\delta^2} \log \left(\frac{\Delta m}{\delta p_\star} \right),$$

where p_\star denotes the smallest entry of the invariant probability vector of G , and where $\gamma(G)$ denotes the spectral gap of the additive reversibilization of G .

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2 Proof of the main result

The proof of Theorem 1 builds upon a simple but fruitful idea, recently introduced by the last two authors in [31]: the *regularization trick*. It consists in restricting functional inequalities to observables $f: \mathcal{X} \rightarrow (0, \infty)$ that are *smooth*, in an appropriate sense.

2.1 The regularization trick

Fix a parameter $r \geq 1$. Following [31], we say that a function $f: \mathcal{X} \rightarrow (0, \infty)$ is *r-regular* if

$$\forall (x, y) \in E, \quad f(x) \leq r f(y). \quad (10)$$

Our starting point is the elementary but new observation that, once restricted to *r-regular* functions, the basic inequality (2) can be reversed at the optimal cost

$$\mathcal{H}(r) := \frac{\sqrt{r} + 1}{\sqrt{r} - 1} \log r. \quad (11)$$

Note that $\mathcal{H}(\infty) = \infty$, in agreement with our earlier observation that the unrestricted inequality (2) can *not* be reversed. At the other extreme, we have $\mathcal{H}(1+) = 4$, in agreement with the fact that (2) is an equality in the infinitely smooth setting of diffusions on manifolds.

Lemma 1 (Exploiting regularity). *If $f: \mathcal{X} \rightarrow (0, \infty)$ is r-regular, then*

$$\mathcal{E}(f, \log f) \leq \mathcal{H}(r) \mathcal{E}(\sqrt{f}, \sqrt{f}).$$

Proof. With $\mathcal{H}(1) := 4$, the formula (11) defines a function $\mathcal{H}: (0, \infty) \rightarrow (0, \infty)$ which increases on $[1, \infty)$ and satisfies $\mathcal{H}(u) = \mathcal{H}(u^{-1})$ for all $u \in (0, \infty)$. It follows that $\mathcal{H}(u) \leq \mathcal{H}(r)$ for all $u \in [r^{-1}, r]$. On the other hand, elementary manipulations give

$$(u - v) \log \left(\frac{u}{v} \right) = \mathcal{H} \left(\frac{u}{v} \right) (\sqrt{u} - \sqrt{v})^2,$$

for all $u, v > 0$. Taking $u = f(x)$ and $v = f(y)$, and recalling our assumption (10), we obtain

$$(f(x) - f(y)) \log \left(\frac{f(x)}{f(y)} \right) \leq \mathcal{H}(r) (\sqrt{f(x)} - \sqrt{f(y)})^2,$$

for any $(x, y) \in E$. To conclude, we multiply by $\pi(x)Q(x, y)$ and sum over all $x, y \in \mathcal{X}$. \square

Let us respectively denote by $t_{\text{LS}}(r)$ and $t_{\text{MLS}}(r)$ the optimal constants in the inequalities (LSI) and (MLSI), when restricted to r -regular functions. The above lemma readily implies

$$t_{\text{LS}}(r) \leq \mathcal{H}(r)t_{\text{MLS}}(r), \quad (12)$$

which can be seen as a regularized version of Theorem 1. To conclude, we now need to relate the constants $t_{\text{MLS}}(r)$ and $t_{\text{LS}}(r)$ to their unregularized versions t_{MLS} and t_{LS} . Of course, we trivially have $t_{\text{MLS}}(r) \leq t_{\text{MLS}}$ and $t_{\text{LS}}(r) \leq t_{\text{LS}}$, by definition. We will now show that those inequalities can be reversed, provided r is large enough. Specifically, we henceforth set

$$r := \frac{4}{p^2}, \quad (13)$$

and we assume that $p \leq 1/2$ (if this is not the case, then irreducibility and reversibility imply that $|\mathcal{X}| \leq 2$, so that Theorem 1 can be checked by hands). Note that we then have $3\mathcal{H}(r) \leq 20 \log \frac{1}{p}$. Thus, Theorem 1 is a consequence of the following crucial result.

Theorem 2 (Regularization). *With r as in (13), we have $t_{\text{LS}} \leq 3t_{\text{LS}}(r)$ and $t_{\text{MLS}} \leq 3t_{\text{MLS}}(r)$.*

In other words, to establish (LSI) or (MLSI) for a reversible Markov chain, it is enough to restrict attention to r -regular observables. Such a costless regularization is of course interesting beyond its role in the present paper. A first version of it (for t_{MLS} , and with an additional factor $\gamma = \frac{\max \pi}{\min \pi}$ in the regularization parameter r) was recently used in [31] to obtain a sharp (MLSI) for the *switch chain* on regular bipartite graphs and as a consequence prove a long-standing conjecture about the mixing time of that chain (see [32] and references therein). Our proof below follows the same strategy, but optimizes it so as to remove the dependency of r on γ . Note that this improvement is crucial for our application to the ZRP.

We write $d(\cdot, \cdot)$ for the graph distance on (\mathcal{X}, E) . Following [31], we fix an arbitrary observable $f: \mathcal{X} \rightarrow (0, \infty)$ and we define a new observable $f_\star: \mathcal{X} \rightarrow (0, \infty)$ by

$$\forall x \in \mathcal{X}, \quad f_\star(x) := \max_{z \in \mathcal{X}} r^{-d(x,z)} f(z).$$

It is immediate to check that f_\star is r -regular and above f (it is in fact the smallest such function, but we will not use this property here). The following propositions show that the quantities $\mathcal{E}(\sqrt{f}, \sqrt{f})$, $\mathcal{E}(f, \log f)$ and $\text{Ent}(f)$ do not change much upon replacing f by f_\star .

Proposition 1 (Comparison of Dirichlet forms). *For r as in (13), we have*

$$\mathcal{E}(\sqrt{f_\star}, \sqrt{f_\star}) \leq \frac{4}{3}\mathcal{E}(\sqrt{f}, \sqrt{f}) \quad \text{and} \quad \mathcal{E}(f_\star, \log f_\star) \leq \frac{4}{3}\mathcal{E}(f, \log f).$$

Proposition 2 (Comparison of entropies). *For r as in (13), we have $\text{Ent}(f) \leq 2\text{Ent}(f_\star)$.*

Those propositions clearly imply Theorem 2, and we henceforth focus on their proofs.

2.2 Comparing Dirichlet forms (Proposition 1)

The proof of Proposition 1 consists of three steps, of which only the last one uses the specific choice for r made at (13). We define the distance from an edge $e \in E$ to $e' \in E$ in the obvious way: $d(e, e') = \ell - 1$, where $\ell \geq 1$ is the minimum length of a path (x_0, \dots, x_ℓ) with $(x_0, x_1) = e$ and $(x_{\ell-1}, x_\ell) = e'$. We first compare the variations of f and f_\star across edges.

Lemma 2. *For each $e = (x, y) \in E$ with $f_\star(x) \leq f_\star(y)$, there is $e' = (x', y') \in E$ so that*

$$r^{-d(e, e')} f(x') \leq f_\star(x) \leq f_\star(y) \leq r^{-d(e, e')} f(y'). \quad (14)$$

Proof. If $f_\star(y) = f(y)$, then we simply choose $e' = e$ and (14) trivially holds. If on the contrary $f_\star(y) < f(y)$, then we take $e' = (x', y')$ where y' is such that $f_\star(y) = r^{-d(y, y')} f(y')$, and where x' is the penultimate vertex on a geodesic from y to y' . We then have $d(e, e') = d(y, y') = 1 + d(y, x') \geq d(x, x')$. Thus, the last inequality in (14) holds with equality, while the definition of f_\star ensures that $f_\star(x) \geq r^{-d(x, x')} f(x') \geq r^{-d(e, e')} f(x')$. \square

We now use this local comparison to estimate how the global quantities $\mathcal{E}(\sqrt{f}, \sqrt{f})$ and $\mathcal{E}(f, \log f)$ change upon replacing f with f_\star . For $(x, y) \in E$, we introduce the short-hand

$$c(x, y) := \pi(x)Q(x, y).$$

Lemma 3. *We have $\mathcal{E}(\sqrt{f_\star}, \sqrt{f_\star}) \leq \kappa \mathcal{E}(\sqrt{f}, \sqrt{f})$ and $\mathcal{E}(f_\star, \log f_\star) \leq \kappa \mathcal{E}(f, \log f)$, where*

$$\kappa := \max_{e' \in E} \left\{ \frac{1}{c(e')} \sum_{e \in E} c(e) r^{-d(e, e')} \right\}. \quad (15)$$

Proof. Let $\nabla f(e) := f(y) - f(x)$ denote the discrete gradient of f across an edge $e = (x, y)$. To each edge $e \in E$, Lemma 2 associates a new edge $e' \in E$ such that

$$\left(\nabla \sqrt{f_\star} \right)_+(e) \leq r^{-d(e, e')} \left(\nabla \sqrt{f} \right)_+(e'),$$

where $h_+ = \max(h, 0)$ denotes the positive part of h . Consequently, we have

$$\begin{aligned} \mathcal{E}(\sqrt{f_\star}, \sqrt{f_\star}) &= \sum_{e \in E} c(e) \left(\nabla \sqrt{f_\star} \right)_+^2(e) \\ &\leq \sum_{e, e' \in E} c(e) r^{-d(e, e')} \left(\nabla \sqrt{f} \right)_+^2(e') \\ &\leq \kappa \sum_{e' \in E} c(e') \left(\nabla \sqrt{f} \right)_+^2(e') = \kappa \mathcal{E}(\sqrt{f}, \sqrt{f}), \end{aligned}$$

as desired. The second claim is obtained in exactly the same way. \square

To obtain Proposition 1, it finally remains to estimate the constant κ defined at (15).

Lemma 4. *Choosing r as in (13) ensures that $\kappa \leq 4/3$.*

Proof. If $e, e' \in E$ satisfy $d(e, e') = 1$, then we can write $e = (x, y)$ and $e' = (y, z)$. Using reversibility and the definition of p at (4), we have

$$c(e) = \pi(x)Q(x, y) = \pi(y)Q(y, x) \leq \pi(y)Q(y) \leq p^{-1}\pi(y)Q(y, z) = p^{-1}c(e').$$

By an immediate induction, we deduce that $c(e) \leq p^{-d(e, e')}c(e')$ for all $e, e' \in E$. Thus,

$$\kappa \leq \max_{e' \in E} \left\{ \sum_{e \in E} (rp)^{-d(e, e')} \right\} \leq \sum_{k=0}^{\infty} \Delta^k (rp)^{-k},$$

where Δ denotes the maximum degree of the graph (\mathcal{X}, E) . But $\Delta \leq p^{-1}$, because from every vertex, the outgoing transition probabilities are at least p and must sum to 1. Thus, $\kappa \leq \sum_k (p^2 r)^{-k} = 4/3$, thanks to our choice for r . \square

2.3 Comparing entropies (Proposition 2)

As in [31], we use the variational characterization of entropy [4]: for any $g: \mathcal{X} \rightarrow (0, \infty)$,

$$\text{Ent}(g) = \max \{ \mathbf{E}[gh] : h \in \mathbb{R}^{\mathcal{X}}, \mathbf{E}[e^h] \leq 1 \}. \quad (16)$$

We start with an elementary lemma, which only uses the fact that $f_{\star} \geq f$.

Lemma 5. *We have $5\text{Ent}(f) \leq 6\text{Ent}(f_{\star}) + (6 \log 6)\mathbf{E}[f_{\star} - f]$.*

Proof. Upon multiplying f (hence also f_{\star}) by a constant, we may assume without loss of generality that $\mathbf{E}[f] = 1$. Then, the function $h := \log\left(\frac{1+5f}{6}\right)$ satisfies $\mathbf{E}[e^h] = 1$ so (16) yields

$$\text{Ent}(f_{\star}) \geq \mathbf{E}[f_{\star}h] = \mathbf{E}[fh] + \mathbf{E}[(f_{\star} - f)h].$$

The claim now readily follows from the pointwise bounds $h \geq \frac{5}{6}\log f$ and $h \geq \log \frac{1}{6}$. \square

In view of this lemma, Proposition 2 boils down to the following result.

Lemma 6. *With r as in (13), we have $(3 \log 6)\mathbf{E}[f_{\star} - f] \leq \text{Ent}(f)$.*

Proof. For each $x \in \mathcal{X}$, let us choose a state $T(x) \in \mathcal{X}$ that achieves the maximum in the definition of $f_\star(x)$ (breaking ties arbitrarily), i.e.

$$f_\star(x) = r^{-d(x, T(x))} f(T(x)).$$

Thus, $T(x) = x$ if and only if $f_\star(x) = f(x)$. Note that if we had $f(T(x)) < f_\star(T(x))$, then

$$f_\star(x) < r^{-d(x, T(x))} f_\star(T(x)) = r^{-d(x, T(x))} r^{-d(T(x), T^2(x))} f(T^2(x)) \leq r^{-d(x, T^2(x))} f(T^2(x)),$$

which would contradict the maximal definition of $f_\star(x)$. Thus, we must in fact have $f(T(x)) = f_\star(T(x))$, or equivalently, $T^2(x) = T(x)$. This shows that $T(A) = A^c$, where

$$A := \{x \in \mathcal{X} : T(x) \neq x\} = \{x \in \mathcal{X} : f(x) \neq f_\star(x)\}.$$

Now, coming back to our goal, let us write

$$\begin{aligned} \mathbf{E}[f_\star - f] &= \sum_{x \in A} \pi(x) (f_\star(x) - f(x)) \\ &= \sum_{x \in A} \pi(x) f(T(x)) r^{-d(x, T(x))} - \sum_{x \in A} \pi(x) f(x) \\ &= \sum_{y \in T(A)} \pi(y) f(y) h(y) - \sum_{x \in A} \pi(x) f(x), \end{aligned}$$

where for $y \in T(A)$, we have introduced the short-hand

$$h(y) := \sum_{x \in T^{-1}(\{y\})} \frac{\pi(x)}{\pi(y)} r^{-d(x, y)}.$$

Recalling that $T(A) = A^c$, we may set $h = -1$ on A to rewrite the previous computation as

$$\mathbf{E}[f_\star - f] = \mathbf{E}[fh].$$

In light of the variational characterization of entropy (16) (with $3h \log 6$ instead of h), it remains to check that $\mathbf{E}[6^{3h}] \leq 1$. As in the proof of Lemma 4, our choice $r = 4p^{-2}$ easily ensures that $h \leq \sum_{k=1}^{\infty} (p^2 r)^{-k} = 1/3$, so that $6^{3h} \leq 1 + 15h$ on A^c . Recalling the definition of h , we deduce that

$$\mathbf{E}[(6^{3h} - 1)\mathbf{1}_{A^c}] \leq 15\mathbf{E}[h\mathbf{1}_{A^c}] = 15 \sum_{x \in A} \pi(x) r^{-d(x, T(x))} \leq \frac{15}{16} \pi(A),$$

because $r \geq 16$. On the other hand, $\mathbf{E}[(6^{3h} - 1)\mathbf{1}_A] = (6^{-3} - 1)\pi(A) \leq -\frac{15}{16}\pi(A)$. \square

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